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Dimensional-duality and Its Lie Groups

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Abstract

For a claim to a dimensional duality, we consider here that, the relativity is depending on a *"double-fold"* complex number for locally dense fourth axis within an enveloping $3D$-space. This dimensional duality has been made here for locally dense m-dimensional geometry within n-space, for $m > n$, if every axis of m-space is dimensional-dual to its enveloping n-space. This locally dense m-dimensional geometry describes a reflexive complex function, viz., *"transfusion"* transformation, which establishes that, Lie group $U(2)$ is the simply connected 1 to 2 enveloping group of $SO(3,1)$ within D-dual spaces only. Again, using the weight vectors, it is found that, there exists a $SU(4)$ group, which *may be* a symmetry group for gravitons.

1 Introduction

If a particle in a Lorentz invariant system is moving with velocity v along its x_3 axis, then it is possible to explain the position vector as a complex number. But, immediately, it arises that this complex number is originally a *"double-fold"* complex number, rather than a conventional complex number. Though, this *"double-fold"* complex number is not a term which is geometrically discussable. From conventional geometry, it is not possible to explain what is the property or meaning of *"double-fold"* in a certain complex number. Here, we are not only very willing to formulate these property and meaning of *"double-fold"* in complex number by developing a new geometrical concept, but also interested to develop a duality of dimensions for locally dense dimensional geometry within its enveloping space due to this *"double-fold"* complex number. Dimensionally, this locally dense geometry is higher dimensional than its enveloping space. Unlike Kaluza-Klein states, we shall consider *no topological extra dimensions* in the way of approaching to this locally dense higher dimensional space. We shall see this geometry can describe that, a Lorentz invariant Lie group $U(2)$ is the simply connected 1 to 2 enveloping group of $SO(3,1)$ within D-dual spaces only. We shall also establish here that, a group $SU(4)$ is emerging from this kind of Dimensional-duality, which *may be* a symmetry for gravitons.

2 Concept of Gravity

Let a particle is moving with velocity v along its x_3 axis. Let, the system is Lorentz invariant. Considering, $\vec{\mathbf{r}} \equiv z$, we have, $|\vec{\mathbf{r}}|^2 \equiv |z|^2 \equiv x_1^2 + x_2^2 + x_3^2 - c^2 \cdot t^2$. Let, z is a complex number, then obviously $z \equiv \langle x_1, x_2, x_3, x_0 \rangle \equiv \sum x_k + ix_0 \equiv x_k + ix_0$, using summation convention, for $k = 1, 2, 3$, implies, $x_0 \neq c \cdot t$ [1].

Definition 1 *Let, the system is Lorentz invariant and since, x_3 is moving with v and, x_3' is being observed at O', where two points O-O' lie in x_3 direction, with a distance $c \cdot t$, then, O-O' is also behaving like a real direction. Let, O-$O' \neq x_3$, where the complex number α is due to O-O'. But, since the axis x_3 and x_3' are coincident permanently, the $\Re(\alpha)$ is always remain as x_k, i.e., $\alpha \equiv x_k + ic \cdot t$. Therefore, z should be written as, $\alpha : z \longrightarrow z \equiv x_k + ix_0$, for $x_0 \equiv ic \cdot t$, i.e., z is not simply a complex number, it is strictly an α dependent complex number, i.e., z is a* "double-fold" *complex number.*

Instead of imaginary part, $\Im(z)$, by introducing, $\mathfrak{D}(z) \equiv x_0$, we may explain the imaginary part of *"double-fold"* complex number z.

Since, $\beta \equiv (v/c)$ is a local property of x_k, due to local Minkowski coordinates, where, $x_k = x_k(\chi)$ in a small neighborhood around a particular event is relevant, then, the total probability of *existence changes in local inertial coordinates for a transformation event* will be unity, i.e., $\varphi + \beta = 1$, where φ is interpreted as an *amplitude of the probability of presence*. If we consider it as a *"double-fold"* complex number, i.e., $\varphi = 1 + i \cdot (i\beta)$, then, it defines the position probability density as, $|\varphi|^2 = 1 - \beta^2$. Then, for the transformation $x_3 \rightarrow x_3'$, the relative position x_3' must be expressed with the amplitude of the probability of presence $|\varphi|$ due to reduction of x_3 by the factor $|\varphi|$ in its direction of motion, i.e., the Lorentz transformation is possible to write as, $x_3 = x_3'|\varphi| + \beta c \cdot t$, and, $c \cdot t = c \cdot t' \, |\varphi| + \beta x_3$, though, these equations are *not* truly Lorentz transformations due to the probabilistic value of $|\varphi|$. Now, we can define gravity as, $\alpha : g_{\mu\nu} \longrightarrow g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, it describes gravity as a universal deformation of Minkowski metric depending on a *"double-fold"* complex number for the fourth axis within a $(3 + 1)D$-space. Similarly, the source of a massless spin-2 field $h_{\mu\nu}$, must then be the α dependent total, conserved energy-momentum tensor [1].

3 Dimensional-duality

The *"double-fold"* complex number, z, is not a term which is geometrically discussable. We don't know yet from conventional geometry, what is the property or meaning of *"double-fold"* in a certain complex number. Here, we shall formulate the property and meaning of *"double-fold"* in complex number by developing a new geometrical concept. The very first thing one can notice in z is that, $x_0 \equiv ic \cdot t \equiv i(c \cdot t) \equiv i \cdot \Im(\alpha)$, is a clear definition of a *"space"* within a *global* or, *enveloping*, space.

Similarly, from Quantum-Gravity wave equation for spatial-spatial coordinates [1], we know,

$$[E - E^{(0)}]\, \mho\,(\vec{X}^{\alpha}, \tau) = -\frac{\hbar^2}{2m}\nabla^2\, \mho\,(\vec{X}^{\alpha}, \tau) \tag{1}$$

where, $E \to i\hbar\frac{\partial}{\partial\tau}|_{\alpha}$, and $E^{(0)} = \vec{p}^{(0)\,2}/(2m)$, using $\vec{p}^{(0)} \to -\frac{\hbar}{c}\frac{\partial}{\partial\tau}$, for $\alpha \equiv (0,1,2,3)$, and $\tau \equiv$ time, which gives the fourth-coordinate as, for some restricted manners, $\vec{X}^0 \equiv \langle ix^{\varepsilon}, ix^{\varepsilon}\rangle^{\frac{1}{2}} \to \vec{X}^0 \equiv \langle ix^{\varepsilon}, ix^{\varepsilon}\rangle^{\frac{1}{2}} \equiv ic\tau$. Since, the spatial-spatial structure get an additional description by considering *"double-fold"* complex number, it is clear from Eq. (1) that, after Lorentz transformation, a particle gains only a certain amount of energy, $E^{(0)}$, within a $3D$ space for its fourth dimension. Since, z is a *"double-fold"* complex number, here, the energy, $E^{(0)}$, is demanding a definition of a *"locally dense"* $(3+1)D$ space within a *global* or, *enveloping*, $3D$ space. Let assume the following proposition:

Proposition 1 *Since, the representation of space must exist as a foundation, and since every conception of space be considered as an infinite multitude of different possible representations, we can arrange the perceptions of infinite representations for certain conception of space by the determinations of the internally dense infinite multitude of sub-representations within that enveloping space representation, exactly in the same manner as we arrange those infinite representations for the external reality of space, if and only if all representations are only parts of one and the same space. Therefore, internally dense different multitudes of sub-representations are not successive but coexistent.*

The Proposition 1 would give neither strict universality, nor apodictic certainty. Let that representation for certain conception of enveloping space be a sphere in a finite-dimensional vector space $\mathcal{V}$ for vectors $\mathbf{x} \equiv x^i \in \mathcal{V}$, and those different multitudes of sub-representations are internally dense within that sphere, precisely, in the upper and/or lower hemisphere of that sphere in $\mathcal{V}$-space.

Let us restrict n-dimensional complex representation to the space $V = \mathbb{C}^n$, then a sphere $S(x_1, \ldots, x_n)$ in $\mathcal{V}$-space gives the total number of 2^n-*ants* (here, -*ant* is extracted from **octant** and intended to mean *"one of* 2^n*-th parts into which n-planes intersecting at point divide space round it"*, and we shall use a common form, *"nant"*, hereafter) for $x_i \in \mathbb{C}^n$, in its upper and lower hemispheres as, $\{(\pm x_{1\,m}, \ldots, \pm x_{n\,m})\,|\,m = 1, \ldots, \frac{2^n}{2}\} = \{\mathcal{O}(x_{1\,M}, \ldots, x_{n\,M})\,|\,M = 1, \ldots, 2^n\}$. Let define separately $\mathcal{O}^{\star}(x_{1\,m}, \ldots, x_{n\,m})$ and $\mathcal{O}_{\star}(x_{1\,m}, \ldots, x_{n\,m})$ for the number of *nants* in upper and lower hemispheres respectively, then, $\mathcal{O}(x_{1\,M}, \ldots, x_{n\,M}) = \{\mathcal{O}^{\star}(x_{1\,m}, \ldots, x_{n\,m}), \mathcal{O}_{\star}(x_{1\,m}, \ldots, x_{n\,m})\,|\,m = 1, \ldots, \frac{2^n}{2}\}$.

Definition 2 *If* $S(p_{1\,N}, \ldots, p_{n\,N})$ *are N sub-spheres locally dense, i.e.,* "one dummy sphere within an enveloping sphere with a common centre of curvature"*, in a sphere* $S(x_1, \ldots, x_n)$ *in* $\mathcal{V}$*-space,* *then the sphere,* $S(x_1, \ldots, x_n)$*, is called a global-sphere (or, enveloping-sphere), and the N sub-spheres,* $S(p_{1N}, \ldots, p_{nN})$*, are called locally dense (dummy) sub-spheres. If the sphere,* $S(x_1, \ldots, x_n)$*, has maximum number of radii for each set of these dummy coordinates,* $(p_{1N}, \ldots, p_{nN})$*, which are equidistant radii in certain* nants *of* $O(x_{1M}, \ldots, x_{nM})$*, or at least in certain planes,* $\{(\chi_1, \ldots, \chi_a) \,|\, \chi_{1,\ldots,a} = \pm x_{1,\ldots,a}; a \in n\}$*, then the N sub-spheres,* $S(p_{1N}, \ldots, p_{nN})$*, are called locally dense maximum-symmetric sub-spheres in the global-sphere* $S(x_1, \ldots, , x_n)$*. On contrary, for minimum number of such equidistant radii for each set of these dummy coordinates,* $(p_{1N}, \ldots, p_{nN})$*, the N sub-spheres,* $S(p_{1N}, \ldots, p_{nN})$*, are called locally dense minimum-symmetric sub-spheres in the global-sphere* $S(x_1, \ldots, x_n)$*.*

Let us define a simplest example. Let p be an equidistant radius in the plane, $(x_1, \ldots, x_a)$, in a global-sphere $S(x_1, \ldots, x_n)$ in $\mathcal{V}$-space. Let p represents itself as a member of $\{\text{card } A : a \in A\}$ orthogonal surfaces lie between $(x_1, \ldots, x_a)$ plane and its neighbouring *nants*. Let the radii q, in $O^\star(x_1, x_2, \ldots, -x_n)$ *nant*, and r, in $O^\star(x_1, -x_2, \ldots, -x_n)$ *nant*, etc., of these $|A|$-orthogonal surfaces are also orthogonal themselves, then, $S(p, q, \ldots, r)$ is a sub-sphere locally dense in the global-sphere $S(x_1, \ldots, x_n)$. Obviously, $p_1 \equiv p$ and $p_n \equiv r$, etc. Note here, neither q and, nor r, etc., are the equidistant radii on their *nants*. In the Figure 1, we have figured a locally dense (dummy) sub-sphere within a global-sphere $S(x_1, x_2, x_3)$.

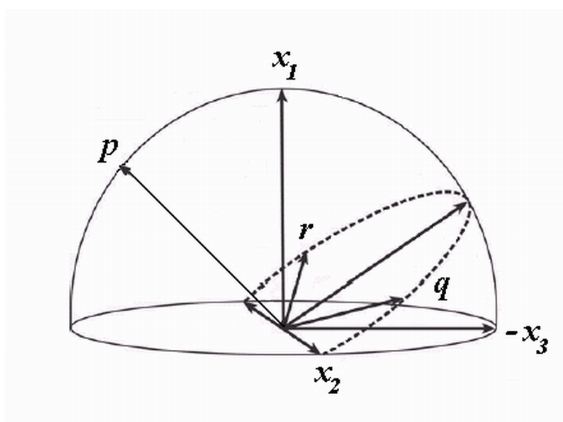

Figure 1: An Example of locally dense (dummy) sub-sphere $S(p, q, r)$ within a global-sphere $S(x_1, x_2, x_3)$.

Theorem 1 *A global-sphere (or, enveloping-sphere)* $S(x_1, \ldots, x_n)$ *in* $\mathcal{V}$*-space, has at most* $l \equiv \{\text{card } O^\star(x_{1m}, \ldots, x_{nm})\}$*, i.e.,* $|O^\star(x_{1m}, \ldots, x_{nm})|$*,*[1] *locally dense maximum-symmetric sub-spheres, if only one radius for each set of these dummy coordinates is an equidistant radius in certain* nant *of* $O^\star(x_{1m}, \ldots, x_{nm})$*, in the global-sphere* $S(x_1, \ldots, x_n)$*.*

Proof. Let an equidistant radius, p_1, in $O^\star(x_1, x_2, \ldots, x_n)$ *nant*, represents itself as a member of $\{\text{card } A : a \in A\}$ orthogonal surfaces lie between $O^\star(x_1, x_2, \ldots, x_n)$ *nant* and its neighbouring

[1] Alternatively, we can use, $\{\text{card } O_\star(x_{1m}, \ldots, x_{nm})\}$.

nants. Let the radii q_1, in $O^\star(x_1, x_2, \dots, -x_n)$ *nant*, and r_1, in $O^\star(x_1, -x_2, \dots, x_n)$ *nant*, etc., of these $|A|$-orthogonal surfaces are also orthogonal themselves, then, $S(p_1, q_1, \dots, r_1)$ is a sub-sphere locally dense in the upper hemisphere of global-sphere $S(x_1, \dots, x_n)$, if and only if neither q_1 and, nor r_1, etc., are the equidistant radii on their *nants*.

Since p_1 is an equidistant of $O^\star(x_1, x_2, \dots, x_n)$, axis x_1 is also an equidistant of *nant* $O^\star(p_1, q_1, \dots, r_1)$. Let an equidistant p_2, in $O^\star(x_1, x_2, \dots, -x_n)$ *nant*, represents another orthogonal set $(p_2, q_2, \dots, r_2)$. Similarly, consider that the equidistant p_l, in $O^\star(x_1, -x_2, \dots, -x_n)$, represents $(p_l, q_l, \dots, r_l)$, and so on. Obviously, $l \equiv \{\text{card } O^\star(x_{1\,m}, \dots, x_{n\,m})\}$, therefore, $l \equiv \frac{2^n}{2}$. It is also obvious that, x_1 is a common equidistant radius of every upper hemisphere dummy *nants*, $\{O^\star(p_k, q_k, \dots, r_k) \,|\, k = 1, \dots, l\}$, of the locally dense dummy sub-spheres $S(p_k, q_k, \dots, r_k)$, by considering each dummy *nant* with *all positive radii*. Since, $p_{k'}$ and $p_{k''}$ are orthogonal, and $p_{k^\#}$ and $p_{k^\$}$, etc., are orthogonal too in pairs, if and only if k' and k'' are odd and, $k^\#$ and $k^\$$ are even, and so on, then the locally dense sub-spheres, $\{S(p_k, q_k, \dots, r_k) \,|\, k = 1, \dots, l\}$ are symmetric maximally in the global-sphere $S(x_1, \dots, x_n)$. □

Corollary 1 *A global-sphere (or, enveloping-sphere) $S(x_1, \dots, x_n)$ in $\mathcal{V}$-space, has at most $\{(l + b) \,|\, b \equiv 0, \frac{l}{2}, \dots,$ for, $n \neq 1\}$ locally dense maximum-symmetric sub-spheres, if only one radius for each set of these dummy coordinates is an equidistant radius in certain plane, $\{(\chi_1, \dots, \chi_a) \,|\, \chi_{1,\dots,a} = \pm x_{1,\dots,a}; a \in n\}$, in the upper hemisphere of global-sphere $S(x_1, \dots, x_n)$.*

Theorem 2 *A global-sphere (or, enveloping-sphere) $S(x_1, \dots, x_n)$ in $\mathcal{V}$-space, has at most $\{(l + b) \,|\, b \equiv 0, \frac{l}{2}, \dots,$ for, $n \neq 1\}$ locally dense maximum-symmetric sub-spheres $\{S(p_k, q_k, \dots, r_k) \,|\, k = 1, \dots, (l + b)\}$, if more than one radius for each set of these dummy coordinates are equidistant radii in certain* nants *of $O(x_{1\,M}, \dots, x_{n\,M})$, in the global-sphere $S(x_1, \dots, x_n)$, and if consider each of its equidistant radii in certain* nants *is a mutual superposition radius of n different p_k and r_k, etc., radii for n different convergence dummy coordinates locally dense in the global-sphere $S(x_1, \dots, x_n)$, i.e., each equidistant radius represents n locally dense different convergence dummy coordinates simultaneously, if and only if each radius of $\{q_k \,|\, k = 1, \dots, (l + b)\}$ is an equidistant radius in certain plane, $\{(\chi_1, \dots, \chi_a) \,|\, \chi_{1,\dots,a} = \pm x_{1,\dots,a}; a \in n\}$.*

Proof. Let the equidistant radius, p_1, in $O(x_1, x_2, \dots, x_n)$ *nant*, represents itself as a member of an orthogonal surface lie between $O(x_1, x_2, \dots, x_n)$ and $O(x_1, -x_2, \dots, -x_n)$ *nants*. Let r_1, in $O(x_1, -x_2, \dots, -x_n)$ *nant*, is an equidistant radius, then another radius q_1, in $O(x_1, x_2, \dots, -x_n)$ *nant*, which is orthogonal both p_1 and r_1, but not an equidistant on $O(x_1, x_2, \dots, -x_n)$, if and only if q_1 is an equidistant on $(x_2, \dots, -x_n)$ plane. Therefore, q_k are equidistant radii in certain planes, $\{(\chi_1, \dots, \chi_a) \,|\, \chi_{1,\dots,a} = \pm x_{1,\dots,a}; a \in n\}$, for $k = 1, \dots, (l + b)$, in general. Since, p_k and r_k, etc., radii are equidistant in certain *nants* of $O(x_{1\,M}, x_{2\,M}, \dots, x_{n\,M})$, in the global-sphere $S(x_1, \dots, x_n)$, then the locally dense sub-spheres, $S(p_k, q_k, \dots, r_k)$, are symmetric maximally in the global-sphere $S(x_1, \dots, x_n)$, which gives each of its equidistant radii in certain *nants* of $O(x_{1\,M}, x_{2\,M}, \dots, x_{n\,M})$, is a mutual superposition radius for n different convergence dummy coordinates, i.e., each equidistant radius is a mutual superposition radius of either p_{k_1}-r_{k_2}-$\dots$-p_{k_n}, etc., or r_{k_1}-p_{k_2}-$\dots$-r_{k_n}, etc., radii for n different convergence dummy coordinates locally dense in the global-sphere $S(x_1, \dots, x_n)$, for $\{k_1 \neq k_2 \neq \ldots \neq k_n \,|\, k = 1, \ldots, (l+b)\}$. Therefore, $\{2(l+b), \ldots\}$ number of p_k-…-r_k, etc., mutual superposition radii for all $(l+b)$ locally dense different convergence dummy coordinates are originally l equidistant radii in $O(x_{1\,M}, x_{2\,M}, \ldots, x_{n\,M})$ in the global-sphere $S(x_1, \ldots, x_n)$. □

Corollary 2 *Let a global-sphere (or, enveloping-sphere) $S(x_1, \ldots, x_n)$ in $\mathcal{V}$-space, has some locally dense maximum-symmetric sub-spheres. If more than one radius of each set of these dummy coordinates are equidistant radii in certain set of planes, $\{(\chi_a, \ldots, \chi_b), (\chi_a, \ldots, \chi_c) \,|\, \chi_{a,\ldots,b,\ldots,c} = \pm x_{a,\ldots,b,\ldots,c}; a, b, c \in n\}$, in the global-sphere $S(x_1, \ldots, x_n)$, then, the global-sphere in $\mathcal{V}$-space has at most $\{(l+b) \,|\, b \equiv 0, \frac{l}{2}, \ldots, \text{for, } n \neq 1\}$ locally dense maximum-symmetric sub-spheres $\{S(p_k, q_k, \ldots, r_k) \,|\, k = 1, \ldots, (l+b)\}$, where each of its equidistant radii in certain set of planes is a mutual superposition radius for different p_k and r_k, etc., radii for different convergence dummy coordinates locally dense in the global-sphere $S(x_1, \ldots, x_n)$, and every member of global-sphere, $\{x_\alpha \,|\, \alpha = 1, \ldots, n\}$, is also a mutual superposition axis with the members $\{q_k \,|\, k = 1, \ldots, (l+b)\}$ of locally dense dummy coordinates in the global-sphere $S(x_1, \ldots, x_n)$.*

Perhaps, it is *unrealistic*, if we consider the representation of *total* dimensions in a certain space is a summation of all locally dense maximum-symmetric sub-space coordinates and its enveloping global-space coordinates. It would be geometrically much *expressive*, if we consider either the summation of all locally dense maximum-symmetric sub-space coordinates in a global-space, or those enveloping global-space coordinates separately for a dimensional representation. The best way of representing the *total* dimensions in a certain space is: *Dimension of locally dense maximum-symmetric sub-space* ⇔ *Dimension of enveloping global-space*, i.e., $\dim(p_{1\,N}, \ldots, p_{n\,N}) \Leftrightarrow \dim(x_1, \ldots, x_n)$, for, $N > n$, in general. Though, it is just a mere representation, however, this enveloping space remains globally unchangeable whether there is any locally dense extra dimension, or not. That is, in both cases, the observed dimensions in laboratory frames always remain fixed at n-dimensions. These extra dimensions are *nothing but the inner property* of this n-dimensional global space. Let us refer this particular type of dual nature transformations as *"Dimensional Duality"*, i.e., *"D-duality"*. Unlike Kaluza-Klein states, for these locally dense maximum-symmetric spaces, we *need not* require to consider any *topological* extra dimensions.

By the way, we may also consider a combined form of Theorem 1 (or Corollary 1) and Theorem 2 (or Corollary 2), or so, for some extremely high dimensional spaces. But this opportunity leads us into the following restriction,

Restriction 1 *For the proper geometrical expressions, let each locally dense maximum-symmetric sub-sphere in a global space in $\mathcal{V}$-space is allowed for either with mutual superposition radii, or without mutual superposition radii only, but not permitted to be attainable both mutual and non-mutual superposition radii simultaneously.*

Since the idea of mutual superposition radii is geometrically quite *meaningless*, then exclusively consider that,

Consideration 3.1 *For a practically useable space, let the* nants *in a global-sphere $S(x_1, \ldots, x_n)$ in $\mathcal{V}$-space, only contain locally dense maximum-symmetric sub-spheres without any mutual superposition radii for these dummy coordinates.*

Theorem 3 *A global-sphere (or, enveloping-sphere) $S(x_1, \ldots, x_n)$ in $\mathcal{V}$-space, has infinite-number of locally dense minimum-symmetric sub-spheres, if almost no radius of each set of these dummy coordinates is an equidistant radius in certain* nant *of $O(x_{1M}, \ldots, x_{nM})$, or in certain plane, $\{(\chi_1, \ldots, \chi_a) \,|\, \chi_{1,\ldots,a} = \pm x_{1,\ldots,a};\, a \in n\}$, in the hemisphere of global-sphere $S(x_1, \ldots, x_n)$.*

Proof. **Case 1.** Decreasing the number of equidistant radii on the *nants* of a certain $\mathcal{V}$-space results increasing the number of locally dense minimum-symmetric sub-spheres. Therefore, vanishing equidistant radii of a $\mathcal{V}$-space results infinitely many locally dense minimum-symmetric sub-spheres.

Case 2. Since, a sphere contains infinite number of radii, and if each of the radii represents itself as a member of a locally dense sub-sphere, then a $\mathcal{V}$-space results infinite-number of locally dense minimum-symmetric sub-spheres. □

Let us define the properties of *D-duality* as:

Property 1 D-duality *has the following properties:*

- D-duality *increases with decrease number of* acting *locally dense maximum-symmetric dimensions, i.e., the enveloping space dimensions result maximum number of locally dense radii at maximum* D-duality *(e.g., Theorem 3). For, $\mathbb{C}^n \to \mathbb{C}^m$, $m > n$, the enveloping space dimensions should be* D-dual *with the locally dense extra dimensions, i.e.,* D-duality *is* **appearing** *to other higher dimensions, e.g., if $\mathbb{C}^m$ is maximum appearing* D-dual *to $\mathbb{C}^n$, $m > n$, then we shall observe the* acting *space has locally dense m dimensions.*

- D-duality *decreases with increase number of* acting *locally dense maximum-symmetric dimensions, i.e., an enveloping space dimensions result minimum number of locally dense* D-dual *coordinates at minimum* D-duality *(e.g., Theorem 1).*

- D-duality *also decreases with* **vanishing** *duality relations upon compactification, i.e., where locally dense maximum-symmetric dimensions are compactified into lower one (e.g., $\mathbb{C}^m \to \mathbb{C}^n$, $m > n$, by using Consideration 3.1 upon Theorem 2 or, Corollary 2). A vanishing* D-duality *of locally dense* acting *dimensions does not mean a complete disappearance of* D-duality *until otherwise it reaches into the enveloping space dimensions, i.e., $\mathbb{C}^n$, where* D-duality *vanishes completely, e.g., if $\mathbb{C}^m$ is maximum vanishing* D-dual *to $\mathbb{C}^n$, $m > n$, then we shall observe the* acting *space has n dimensions, i.e., the global dimensions.*

Dropping the relation *maximum* from the appearing/vanishing *D-dualities*, we find that the *acting* space not necessarily have *m*, or *n* dimensions respectively.

A locally dense m-dimensional geometry within an enveloping $\mathcal{V}$-space, where every axis of these locally dense spaces comes from a dimensional many-one-onto function (for maximum-symmetric locally dense sub-spheres), *gave off* by its parental $\mathcal{V}$-space, is a sufficient background to determine, $\mathbb{C}^m \Leftrightarrow \mathbb{C}^n$, $m > n$, which implies a very important theorem as follows.

Theorem 4 $\mathbb{C} \ \tilde{\Longleftrightarrow}\ \mathbb{R}$.[2]

Proof. We know that, $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$, i.e., $\mathbb{C} \cong \mathbb{R}^2$.

Case 1: $\mathbb{C}^n \Rightarrow \mathbb{C}^m$**:** Let define finite dimensional $\mathcal{V}$-space $\mathbb{C}^n$, and $H \subset \mathbb{C}^n$, where $H = \{O(x_1^1, \ldots, x_n^1), \ldots, O(x_1^a, \ldots, x_n^a) \,|\, a > n\}$ is the set of *nants* in sphere $S(x_1, \ldots, x_n)$. Evidently, $j : H \to \mathbb{C}^n$. Since, H is countable, there exists a many-one-onto function $g : H \to G$, where $G = \{p_1, \ldots, p_n\}$, is the set of radii for a locally dense sub-sphere $S(p_1, \ldots, p_n)$, and let $\dim\ \mathbb{C}^n = \dim\ G$. Then G is countable.

Since $a > n$, then, there should be more than one locally dense sub-spheres $\{S(p_1^b, \ldots, p_n^b) \,|\, b = 1, \ldots, A : A \geq a\}$ within enveloping sphere $S(x_1, \ldots, x_n)$. Therefore,

$$f \circ j : H \to G \times \ldots \times G \,|\, (f \circ j)(h) = f(j(h)) \ \ \forall\, h \in H$$

$$\Rightarrow f \circ j : H \to G \times \ldots \times G \,|\, (f \circ j)(h) = f(\mathbb{C}^n) \ \ \forall\, h \in H.$$

Evidently, f is many-one-into. But $\mathbb{C}^n$ is finite dimensional. Hence, $G \times \ldots \times G$ is countable. Since, $\dim\ \mathbb{C}^n = \dim\ G$, let, $G \times \ldots \times G = \mathbb{C}^m$, $m > n$. Therefore, $\mathbb{C}^m$ is finite dimensional too. Hence, $\mathbb{C}^n \Rightarrow \mathbb{C}^m$.

Since, $\mathbb{C} \cong \mathbb{R}^2$, let, $n = \frac{1}{2}$ and $m = 1$. Then, for $\mathbb{C}^{\frac{1}{2}} \cong \mathbb{R}$, let, $\mathbb{C}^{\frac{1}{2}} \Rightarrow \mathbb{C}$, which gives, at least, $\mathbb{R} \Rightarrow \mathbb{R}^2$, since $\mathbb{C}^n \Rightarrow \mathbb{C}^m$, $m > n$.

Case 2: $\mathbb{C}^m \Rightarrow \mathbb{C}^n$**:** Let $G = P_1 \times \ldots \times P_n$, where $\{p_j \in P_i \,|\, i = j; i, j = 1, \ldots, n\}$ are equidistant and non-equidistant radii of G. Since, $\{P_i^b \subset \mathbb{C}^m \,|\, b = 1, \ldots, A : A \geq a\}$, then there exists,

$$f^{-1} : G \times \ldots \times G \to j(H) \,|\, (f^{-1}(g \times \ldots \times g)) = f^{-1}(\mathbb{C}^m) \ \ \forall\, g \in G.$$

Here, f^{-1} is a many-one-onto function. But $\mathbb{C}^m$ is finite dimensional, and $j(H) = \mathbb{C}^n$, hence, $\mathbb{C}^m \Rightarrow \mathbb{C}^n$. Therefore, $\mathbb{C}^m \Leftrightarrow \mathbb{C}^n$.

Again, for $n = \frac{1}{2}$ and $m = 1$, we have, $\mathbb{C} \Rightarrow \mathbb{C}^{\frac{1}{2}}$, which gives, at least, $\mathbb{R}^2 \Rightarrow \mathbb{R}$, since $\mathbb{C}^m \Rightarrow \mathbb{C}^n$, $m > n$. Therefore, $\mathbb{R}^2 \Leftrightarrow \mathbb{R}$. But, we considered, $\mathbb{C} \cong \mathbb{R}^2$, hence, $\mathbb{C} \ \tilde{\Longleftrightarrow}\ \mathbb{R}$, since $\mathbb{C}^m \Leftrightarrow \mathbb{C}^n$, $m > n$, only within D-dual spaces. □

Restriction 2 *In Theorem 4, it is defined that,* $\mathbb{C}$ *is* D-dual *to* $\mathbb{R}$*. But, Theorem 4 is* only and even only *applicable within a locally dense space, not for any ordinary (non-dense) space, such as Euclidian, Riemannian, etc. The consideration of* $\mathbb{C} \ \tilde{\Longleftrightarrow}\ \mathbb{R}$ *should completely be erroneous within a* non-dense *space, e.g., Euclidian, Riemannian, etc.*

[2] Do not misunderstand this equation as, $\mathbb{C} \Leftrightarrow \mathbb{R}$. The meaning of $\tilde{\Longleftrightarrow}$ is *"different"* from $\Leftrightarrow$ anyway. This Theorem is only applicable within D-dual spaces, but not **elsewhere**, or **anywhere**.

It is *not* a strict rule that, the n-dimensional enveloping space is always complex, i.e., $V = \mathbb{C}^n$, it should simply be a real one too, i.e., $V = \mathbb{R}^n$, and unobjectionably, it also gives the Theorem 4 properly in the form, $\mathbb{R}^2 \Leftrightarrow \mathbb{R}$, therefore, $\left\{ \begin{array}{ccc} \mathbb{C} & \tilde{\Longleftrightarrow} & \mathbb{R} \\ \mathbb{R}^2 & \Leftrightarrow & \end{array} \right\}$, which intends the *universality* of Theorem 4 in this particular form.

4 Lie Groups from Theorem 4

Since, the system of a particle moving with velocity v along its x_3 axis is Lorentz invariant, as we discussed in Section 2, let us consider that, x_3 is satisfying Theorem 4. Then obviously $z \equiv \langle x_1, x_2, x_3, x_0 \rangle \equiv \sum x_k + ix_0 \equiv x_k + ix_0$, using summation convention, for $k = 1, 2, 3$, implies, both x_3 and O-O' are satisfying Theorem 4 individually since, O-$O' \neq x_3$. Therefore, z would be written as, $\alpha : z \to z \equiv x_k + ix_0$, for $\mathfrak{I}(z) \equiv x_0 \equiv \langle ix^\varepsilon, ix^\varepsilon \rangle^{\frac{1}{2}} \equiv ic \cdot t$, for some restricted manners, i.e., z is a *"double-fold"* complex number for $\alpha \equiv x_k + ic \cdot t$, where α is locally dense complex number within the enveloping axis, i.e., x_3, i.e., α is D-dual to x_3 and z is a D-dual *"double-fold"* complex number to x_3.

Definition 3 *Let, z is a* "double-fold" *complex number for α, and $f(\gamma) = \alpha$, then a complex function may call a* "reflexive complex function" *if, $R(\mathbb{C}) : f(\gamma) \to f'(\gamma)$, gives, $f'(\gamma) = z$.*

Lemma 1 *If a function, $f(\gamma)$, transforms into a reflexive complex function, $R(\mathbb{C}) : f(\gamma) \to f'(\gamma)$, for $f'(\gamma) \in \mathbb{C}$ and $f(\gamma) \in \mathbb{A}$, where either $\mathbb{R} \subset \mathbb{A}$ or $\mathbb{C} \subset \mathbb{A}$ depending upon $f(\gamma)$, then, the transformation, $R(\mathbb{C}) : f(\gamma) \to f'(\gamma)$, may call as a* "transfusion" *of $f(\gamma)$ into $f'(\gamma)$. There would be no* "transfusion" *transformation of $f(\gamma)$ into $f'(\gamma)$ for $f'(\gamma) \in \mathbb{R}$, i.e., there exists no reflexive real function, $R(\mathbb{R}) : f(\gamma) \to f'(\gamma)$, for $f(\gamma) \in \mathbb{A}$, where $\mathbb{A}$ should be real or complex.*

Proof. Let, z is a *"double-fold"* complex number for α. Since, $R(\mathbb{C})$ is a reflexive complex function then, $\{x_i, y_j\} \in \mathbb{A} \mapsto (x_i, y_j) \in \mathbb{C}$, where, $\{x_i, y_j\}$ should be $\mathbb{R}$ or $\mathbb{C}$ depending upon α, hence, $\mathfrak{R}(z) \equiv x_i$ and $\mathfrak{I}(z) \equiv y_j$. Therefore, $R(\mathbb{C}) : \alpha \to z$, is the *"transfusion"* transformation of α into z. If there exists a certain transformation $\alpha : z \to z \equiv x_k + ix_l$, for $\alpha \in \mathbb{R}$, then z satisfies a reflexive complex function, $R(\mathbb{R}) : f(\gamma) \to f'(\gamma)$, where z is simply a complex number, rather than a *"double-fold"* complex number. But, for $f'(\gamma) \in \mathbb{R}$, there exists no function, $R(\mathbb{R}) : f(\gamma) \to f'(\gamma)$, which is *"reflexive"*, for $f(\gamma) \in \mathbb{A}$, which satisfying either $\mathbb{R} \subset \mathbb{A}$ or $\mathbb{C} \subset \mathbb{A}$ depending upon $f(\gamma)$. □

Let, $z \equiv x_k + i(ix_l) \equiv x_k - x_l$, and $\alpha \equiv x_k + ix_l$, for $x_l \equiv \langle x^\varepsilon, x^\varepsilon \rangle^{\frac{1}{2}} \equiv c \cdot t$, for some restricted manners, and if, $\mathbf{R}$ transforms $P[z, \alpha]$ into a new $P'[z', \alpha']$, then $P' = \mathbf{R}P$, let the operator, $\mathbf{R}$, is a rotation operator, $\mathbf{R}(\phi)$, in the plane. Let the angle of rotation be sufficiently small for us to put $\cos(\phi) \cong 1$, and $\sin(\phi) \cong \delta\phi$, in which case, we have, $\mathbf{R}(\delta\phi) = \mathbf{I} + d\mathbf{R}(\phi)/d\phi\Big|_{\phi=0} \cdot \delta\phi$, where, $z \equiv x_k + \delta z'$, for $\delta z' = -x_l\delta\phi$, and $\alpha \equiv x_k + \delta\alpha'$, for $\delta\alpha' = ix_l\delta\phi$. Therefore, $\mathbf{R}(\delta\phi)$ involves the infinitesimal Lie transformation.

For $4D$ rotation, the total transformation is, $\mathbf{R}(\phi, \varphi, \chi, \psi) = \mathbf{R}(\phi)\mathbf{R}(\varphi)\mathbf{R}(\chi)\mathbf{R}(\psi)$, then, to first-order in the δ's, we have, $\mathbf{R}(\delta\phi, \delta\varphi, \delta\chi, \delta\psi) = \mathbf{I} + \mathbf{Y}_0\delta\phi + \mathbf{Y}_1\delta\varphi + \mathbf{Y}_2\delta\chi + \mathbf{Y}_3\delta\psi$. Introducing angular momentum operator, $\vec{J}_\kappa = -i\mathbf{Y}_\kappa$, for $\hbar = 1$, $\kappa = 0, 1, 2, 3$, we have the general commutation relations as,

$$\Big[[\vec{J}_0, \vec{J}_1], \vec{J}_2\Big] = i\,\vec{J}_3, \qquad \Big[[\vec{J}_2, \vec{J}_3], \vec{J}_0\Big] = i\,\vec{J}_1, \qquad \Big[[\vec{J}_0, \vec{J}_3], \vec{J}_1\Big] = i\,\vec{J}_2, \qquad \Big[[\vec{J}_i, \vec{J}_j], \vec{J}_\kappa\Big] = i\,\vec{J}_0,$$

for $i \neq j \neq \kappa$, where, $(i, j, \kappa)^\top = (1, 2, 3)$, i.e., the angular momentum operators form a Lie algebra. If they satisfies,

$$\vec{J}_+ = \vec{J}_1 + \vec{J}_0 + i\vec{J}_2, \qquad \vec{J}_- = \vec{J}_1 + \vec{J}_0 - i\vec{J}_2, \qquad \vec{J}_1 = \tfrac{1}{2}(\vec{J}_+ + \vec{J}_- - 2\vec{J}_0),$$

$$\vec{J}_0 = \tfrac{1}{2}(\vec{J}_+ + \vec{J}_- - 2\vec{J}_1), \qquad \vec{J}_2 = \tfrac{1}{2i}(\vec{J}_+ - \vec{J}_-),$$

then, we have the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From another point of view, for rotations [2] in 4-dimensional space, we can write for the position vector $\vec{r}$ which rotating through the small angle $d\omega = |d\omega|$ about an axis in the direction of the vector $d\omega$, as:

$$\vec{r}_{12} - \vec{r}_{23} - \vec{r}_{31} = (d\omega_1 \times d\omega_2 \times d\omega_3) \times \vec{r}.$$

It is clear that this expression must gives,

$$d\omega_1 \cdot \vec{L}\, d\omega_2 \cdot \vec{L}\, d\omega_3 \cdot \vec{L} - d\omega_2 \cdot \vec{L}\, d\omega_3 \cdot \vec{L}\, d\omega_1 \cdot \vec{L} - d\omega_3 \cdot \vec{L}\, d\omega_1 \cdot \vec{L}\, d\omega_2 \cdot \vec{L} = i(d\omega_1 \times d\omega_2 \times d\omega_3) \cdot \vec{L}.$$

Thus, adopting angular momentum operator $\vec{J}$ for more general point of view, we have the same definitions cited above. Therefore, the fourth matrix is a *"natural object"* in 4-dimensional space.

Remark 1 ***D-dual Enveloping Group:*** *In D-dual enveloping space, the link between groups have to written in reverse order. We know, the Lie group $SU(2)$ is the simply connected 2 to 1 covering group of $SO(3)$, but in a D-dual space, we start from $SO(3)$ and finally reach at $SU(2)$, therefore, in a D-dual space the Lie group $SO(3)$ is the simple connected 1 to 2 D-dual (globally) enveloping group of $SU(2)$. That is,* "D-dual enveloping" *group is completely different from our commonly used* idea *of covering groups.*

Theorem 5 *If the* "double-fold" *complex number, z, gives a mapping, $\langle z, \mathbb{C}\rangle \mapsto \langle |z(\mathbb{C})|, \mathbb{R}\rangle$, then the Lie group $U(2)_z$ is the simply connected 1 to 2 enveloping group of $SO(3,1)$, for a Lie group* "transfusion" *transformation, $R(\mathbb{C}) : U(2)_\alpha \to U(2)_z$ within D-dual spaces only.*

Proof. Due to the form of *"double-fold"* complex number, $z \equiv x_k + ix_0$, where $k = 1, 2, 3$ for $\alpha \equiv \langle x^\beta, x^\beta\rangle^{\frac{1}{2}} + \langle ix^\varepsilon, ix^\varepsilon\rangle^{\frac{1}{2}} \equiv x_k + ic\cdot t$, since $x_0 \equiv \langle ix^\varepsilon, ix^\varepsilon\rangle^{\frac{1}{2}} \equiv ic\cdot t$, for some restricted manners, we can write a Lie group *"transfusion"* transformation of $U(2)_\alpha$ into $U(2)_z$ as, $R(\mathbb{C}) : U(2)_\alpha \to U(2)_z$. Since, $z \equiv x_k + ix_0 \mapsto |z(\mathbb{C})| \equiv \sqrt{x^k x_k - x^l x_l}$, for $x_l \equiv \langle x^\varepsilon, x^\varepsilon \rangle^{\frac{1}{2}} \equiv c \cdot t \in \mathbb{R}$, and, $x_0 \equiv ix_l \in \mathbb{C}$, we have, $\langle z, \mathbb{C} \rangle \mapsto \langle |z(\mathbb{C})|, \mathbb{R} \rangle$. If, $\mathfrak{a} \in \mathfrak{u}(2)$, then, $(T_e\pi(\mathfrak{a})\boldsymbol{x}) \cdot \boldsymbol{\sigma} = (\mathfrak{a}\ \boldsymbol{x}) \cdot \boldsymbol{\sigma}$, where, $\boldsymbol{\sigma}$ be the Pauli spin matrices. But, $\langle \mathfrak{a}, \mathbb{C} \rangle \mapsto \langle |\mathfrak{a}(\mathbb{C})|, \mathbb{R} \rangle$, thus, $T_e\pi : \mathfrak{gl}(2, \mathbb{C}) \to \mathfrak{gl}(4, \mathbb{R})$. Therefore, $\pi : U(2) \to SO(3,1)$ within D-dual spaces only. Elsewhere, e.g., within Euclidian, Riemannian, etc., spaces, $\pi : U(2) \nrightarrow SO(3,1)$. □

Definition 4 *If $z \in \mathbb{C}$, then a mapping, $z \mapsto$* **Comp**(z)*, may call a* "component mapping"*, if there exists,* $\{\Re(z), \mathfrak{D}(z)\} \in$ **Comp**(z)*. If $r \in \mathbb{R}$, then a* "component mapping"*, $r \equiv \sum_j m_j \mapsto$* **Comp**$(r)$*, should give, $m_j \in$* **Comp**(r)*.*

Theorem 6 *If the* "double-fold" *complex number, z, gives a mapping, $\langle z, \mathbb{C} \rangle \mapsto \langle |dz(\mathbb{C})|, \mathbb{R} \rangle$, then for a component mapping, $\partial z(\mathbb{C}) \mapsto$* **Comp**$[\partial z(\mathbb{C})]$*, we have the first non-vanishing order of* **Comp**$[\partial z(\mathbb{C})]$ *from the transformation,* **Comp**$[\partial z(\mathbb{C})] : M \to e^\eta_\mu[$**Comp**$(M)] \cdot$ **Comp**(∂M)*, where,*
(1) *M is the* dim z *manifold with tangent indices μ,*
(2) **Comp**$[z(\mathbb{C})]$ *around a particular event A at* **Comp**(M) *with $\alpha \mapsto$* dim z*, and*
(3) $e^\eta_\mu[$**Comp**$(M)]$ *is the Jacobian matrix of the change of μ from* **Comp**(M) *to η of* **Comp**(z)*.*

Proof. Since, M is the dim z manifold with tangent indices μ, and **Comp**$[z(\mathbb{C})]$ around a particular event A at **Comp**(M) with $\alpha \mapsto$ dim z, only the value of **Comp**$[z(\mathbb{C})]$ in a small neighborhood around A is relevant, therefore, the first non-vanishing order of **Comp**$[\partial z(\mathbb{C})]$ we have, **Comp**$[\partial z(\mathbb{C})] = e^\eta_\mu[$**Comp**$(M)] \cdot$ **Comp**(∂M). □

Here, **Comp**$[\partial z(\mathbb{C})] = e^\eta_\mu[$**Comp**$(M)] \cdot$ **Comp**(∂M) is nothing but the expression, $\partial X^\eta(x) = e^\eta_\mu(x)\,\partial x^\mu$, i.e., $e^\eta_\mu[$**Comp**$(M)]$ is the gravitational field, a.k.a., the *tetrad* field, at **Comp**(M).

Theorem 7 *For $U(2)_z$, let the eigenvalue equations, $\lambda_j\mathbf{u} = \alpha_j\mathbf{u}$, where λ_j are diagonal. Since, the weight vectors are formed from the pairs of eigenvalues, $[\alpha_i, \alpha_j]$, then for a mapping, $U(2)_z \mapsto U(2)_z \otimes \overline{U}(2)_z \equiv U(4)$, where $\overline{U}(2)_z$ is the conjugate representation, there exists, $U(4) \mapsto SU(4)$, since,* tr T *is an invariant for the $2_z \otimes \overline{2}_z$ tensor $T^\alpha_\beta = g^\alpha\ \overline{g}_\beta$, which lefts with a traceless tensor, ${T'}^{\alpha}_{\beta}$.*

Proof. Since, $z \in \mathbb{C}$, and the weight vectors formed from the pairs of eigenvalues, $[\alpha_i, \alpha_j]$, then there be a mapping, $U(2)_z \mapsto U(2)_z \otimes \overline{U}(2)_z \equiv U(4)$, for the $2_z \otimes \overline{2}_z$ tensor $T^\alpha_\beta = g^\alpha\ \overline{g}_\beta$. But, tr T is an invariant, and since, we left with a traceless and irreducible tensor, ${T'}^{\alpha}_{\beta}$, there must exists a mapping, $U(4) \mapsto SU(4)$. □

We can write the vector product as, $U(4) \equiv U(2)_z \otimes \overline{U}(2)_z = g^\alpha\ \overline{g}_\beta \sin\theta\ \hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane spanned by $U(2)_z$ and $\overline{U}(2)_z$. Therefore, $U(2)_z \otimes \overline{U}(2)_z$ is a *vector potential*, thus, *may be*, the emerging symmetry, $SU(4)$, is the symmetry for field particles of gravity due to ${T'}^{\alpha}_{\beta}$. Hence, the $U(4) \mapsto SU(4)$ symmetry *may be* a symmetry for gravitons. It is a simple curiosity that, gravitons *do not* observe yet with so many verities. It is true. But what we have here is that, these $15 \oplus 1$ components are emerging from the real axes of $2_z \otimes \overline{2}_z$, which forces us for a further investigation apart from Theorem 7 [3].

5 Discussion

Considering, z is strictly an α dependent complex number, i.e., a *"double-fold"* complex number, where both z and α are locally dense complex numbers within their enveloping axis, x_3, and further considering that, α transforms into a reflexive complex function, *"transfusion"* transformation of α into z, for $z \in \mathbb{C}$ and $\alpha \in \mathbb{A}$, satisfying either $\mathbb{R} \subset \mathbb{A}$ or $\mathbb{C} \subset \mathbb{A}$ depending upon α, then, for a mapping, $\langle z, \mathbb{C}\rangle \mapsto \langle z(\mathbb{C}), \mathbb{R}\rangle$, we have that, Lie group $U(2)_z$ is the simply connected 1 to 2 enveloping group of $SO(3,1)$ within D-dual spaces only. Introducing a component mapping, $z(\mathbb{C}) \mapsto \mathbf{Comp}[z(\mathbb{C})]$, we have the gravitational field, $e^{\alpha}_{\mu}[\mathbf{Comp}(M)]$, from the first non-vanishing order of $\mathbf{Comp}[\partial z(\mathbb{C})]$. Again, using the weight vectors, it is found that, there exists a symmetry $SU(4)$, which *may be* a symmetry for gravitons. Therefore, the picture from Theorem 7 is more complex than simply defining gravity as, $\alpha : g_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, which describes gravity as a universal deformation of Minkowski metric depending on a *"double-fold"* complex number for the locally dense fourth axis within an enveloping $3D$-space, whereas, this simpler form of α dependent source of a massless spin-2 field $h_{\mu\nu}$ is *not* too much expressive than Theorem 7.

Again, in Eq. (1), the energy, $E^{(0)}$, is a definition of a locally dense $(3 + 1)D$ space within a global or, enveloping, $3D$ space. This idea of ***Sujata Relativity*** is quite different from the Einstein, where, he had considered that the time axis is an independent axis [4]-[5] beyond *"real"* space, whereas, by considering *"double-fold"* complex number, Eq. (1) is very near to the real (particle) world.

In Classical Relativity (i.e., Einsteinian Relativity), neither Theorem 5, and nor Theorem 7 are achievable. Dimensional-duality for locally dense complex numbers, z and α, gave a sufficient background for both of these theorems within their enveloping axis, x_3.

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